NUMERICAL SOLUTION TO UPLIFT PRESSURE UNDER A WEIR

1.1 General

Weirs on permeable foundation are designed to safeguard against uplift pressure and piping. The flow characteristics are determined assuming the flow to be two dimensional and steady.

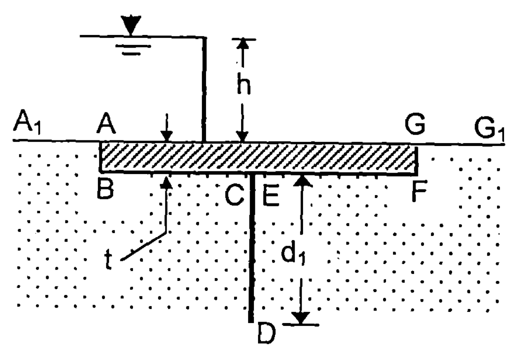
For non-homogeneous sub-soil, numerical method is used to solve the two-dimensional equation satisfying the boundary conditions.

$$\frac{\delta}{\delta x}\left\{-k(x,y)\frac{\delta h}{\delta x}\right\}+\frac{\delta}{\delta y}\left\{-k(x,y)\frac{\delta h}{\delta x}\right\}=0$$

For homogeneous, isotropic soil, the governing equation is the Laplace equation, which is solved analytically using conformal mapping.

Khosla et.al. (1936) found solutions to two-dimensional steady flow under a number of simple profiles of weirs resting on a homogeneous and isotropic soil of infinite depth using the Scwarz-Christoffel conformal transformation technique.

In Khosla's method of analysis, the excess pressure head has been derived assuming the thickness of floor is negligible and the structure is resting on the surface. As the foundation has some thickness, a part of the seepage head is lost along the foundation depth which has to be accounted for. In order to adjust this head loss, Khosla et.al has suggested a correction computed by interpolation under the assumption that, the variation of hydraulic head is linear along the sheet-pile depth and the rate of variation is equal to the variation along the depth of depression.



It may be noted that, the nature of dissipation of head along the depth of depression and sheet-pile are not similar. However, the Pressure head at key points in excess of the hydrostatic head at the downstream boundary have been presented as a percentage of the seepage head in the form of charts effecting some correction considering the depression. In this process they neglected the depth of depression to reduce the number of vertices taking part in the conformal mapping. By reducing the number of vertices, it was possible to carry out the integration required in solving the transformation.

Analytical solutions for a stepped-depressed weir are not available. Current practices involve applying of corrections to the solution that obtained by neglecting depression. As the derived equations of the uplift pressure, exit gradient are highly non-linear and contain elliptic integral of third kind, a numerical integration is necessary in case of structures having vertices more than three. This paper uses the conformal mapping and Newton-Raphson technique for solving the non-linear equation thereby solving the equations of the uplift pressure, exit gradient to a finite end so that no further corrections need to apply over the result.

2. Statement of the Problem

The depressed-stepped weir with a sheet-pile at the step is shown in figure-1. The width of floor upstream to the sheet-pile is '$\mathbf{b_1}$'. The width of the down stream floor is '$\mathbf{b_2}$'. The depth of sheet-pile is '**s**'. The depths of depression of the floor in the upstream, at the junction of sheet-pile and floors; and downstream floor are $\mathbf{d_1}$, $\mathbf{d_2}$ and $\mathbf{d_3}$ respectively.

The heights of water above the upstream and downstream floor bed are $\mathbf{h_1}$ and $\mathbf{h_2}$ respectively and the difference in the total heads between the upstream and downstream boundaries is '**h**'.

3. Analysis

3.1 Mapping of z-plane on to t-plane: z = f(t)

The conformal mapping of the flow domain in z-plane onto the lower half of the auxiliary t-plane is given by:

$$Z = M \int \frac{(m-t)\sqrt{(t+\gamma)(\lambda-t)}}{\sqrt{(t+\beta)(1-t^2)(\mu-t)}} dt + N \qquad \text{---- (3.1)}$$

Please refer Figure 3.1(a) and 3.1(b)in the page following. The vertices A_1, A, B, C, D. E, D, G and G_1 being mapped onto - ∞, β, γ, -1, m, 1, λ, μ and ∞ respectively in the t-plane. M and N are the complex constants to be determined. The constant N is governed by the lower limit of integration. To find the constants M and N, and the relationship between the transformation parameters and the dimension of the structure, integration between the consecutive vertices is carried out as under:

z-plane

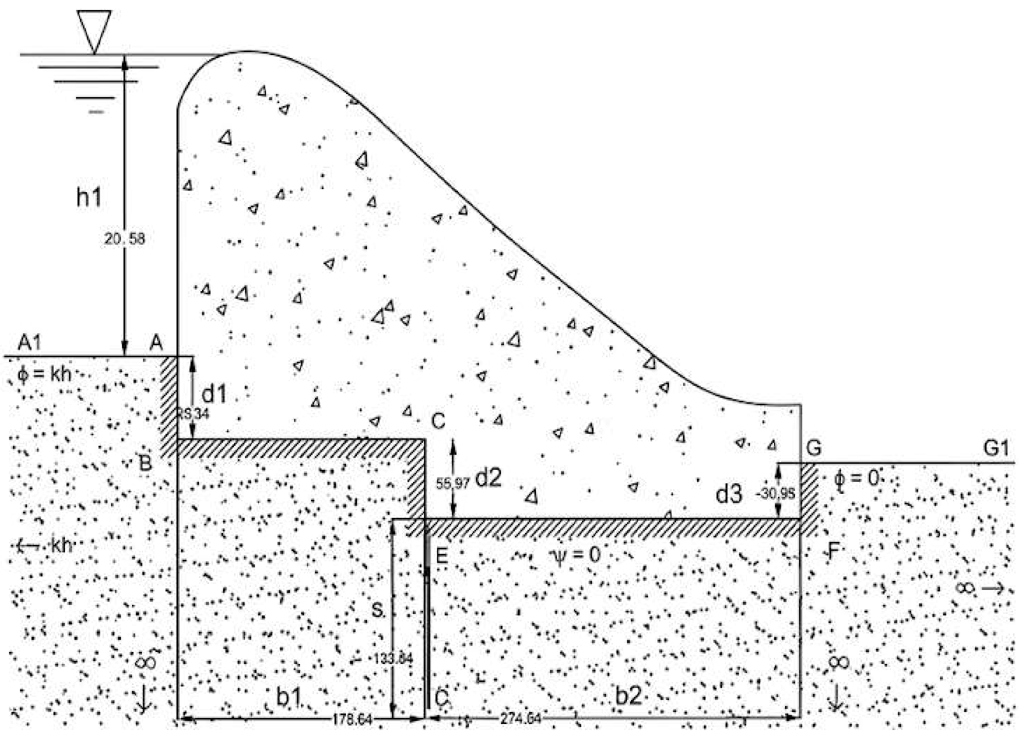


Figure. 3.1(a) : Physical Domain in z-plane

t-plane

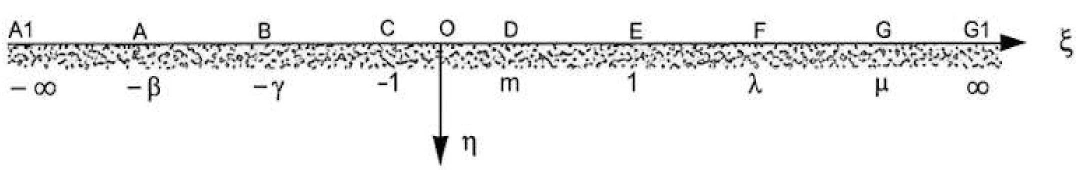


Figure.3.1(b) : Physical Domain Mapped onto t-plane

i. Integration between the vertices D and E ($m \le t \le 1$)

applying the conditions:

at point D: t = m, for which z = -i (s + d_2) and
at point E: t = 1, for which z = id_2
we obtain:

$$-id_2 = M\int_m^1 \frac{(m-t)\sqrt{(t+\gamma)(\lambda-t)}}{\sqrt{(t+\beta)(1-t^2)(\mu-t)}}dt - i\,(s+d_2)$$

or $$-id_2 = M\int_m^1 f_1(t)dt - i(s + d_2)$$

or $$is = M\,I_1 \qquad \text{where } I_1 = \int_m^1 f_1(t)dt$$

or $$M = \frac{is}{I_1} \qquad \text{- - - - (3.2)}$$

ii. Integration between the vertices E and F ($1 \le t \le \lambda$)

applying the conditions:

at point E: t = 1, for which z = -id_2 and
at point F: t = λ, for which z = $b_2 - id_2$
we obtain:

$$b_2 - id_2 = M\int_1^\lambda \frac{(t-m)\sqrt{(t+\gamma)(\lambda-t)}}{\sqrt{(t+\beta)(t^2-1)(\mu-t)}}\frac{dt}{\sqrt{-1}} - i\,d_2$$

or $$b_2 = M\int_1^\lambda f_2(t)dt$$

or $$b_2 = \mathrm{M}\frac{I_2}{\sqrt{-1}} = \frac{is}{I_1}\cdot\frac{I_2}{i} \qquad \text{where } I_2 = \int_1^\lambda f_2(t)dt$$

or $$\frac{b_2}{s} = \frac{I_2}{I_1}$$

or $$\boldsymbol{F_1} = \frac{\boldsymbol{b_2}}{\boldsymbol{s}} - \frac{\boldsymbol{I_2}}{\boldsymbol{I_1}} \qquad \text{- - - - (3.3)}$$

iii. Integration between the vertices F and G ($\lambda \leq t \leq \mu$)

applying the conditions:

at point F: $t = \lambda$, for which $z = b_2 - id_2$ and
at point G: $t = \mu$, for which $z = b_2 - i(d_2 - d_3)$
we obtain:

$$b_2 - i(d_2 - d_3) = M \int_{\lambda}^{\mu} \frac{(t-m)\sqrt{(t+\gamma)(t-\lambda)}}{\sqrt{(t+\beta)(t^2-1)(\mu-t)}} dt + b_2 - i\, d_2$$

or $b_2 - i(d_2 - d_3) = M \int_{\lambda}^{\mu} f_3(t)dt + b_2 - i\, d_2$

or b_2-id_2- $id_3 = MI_3 + b_2$- $i\, d_2$ where $I_3 = \int_{\lambda}^{\mu} f_3(t)dt$

or $\frac{d_3}{s} = \frac{I_3}{I_1}$

or $\boldsymbol{F_2} = \frac{\boldsymbol{d_3}}{\boldsymbol{s}} - \frac{\boldsymbol{I_3}}{\boldsymbol{I_1}}$ ---- (3.4)

iv. Integration between the vertices D and C ($-1 \leq t \leq m$)

applying the conditions:

at point D: $t = m$, for which $z = -i(s + d_2)$ and
at point C: $t = -1$, for which $z = 0$
we obtain:

$$-i(s + d_2) = M \int_{-1}^{m} \frac{(m-t)\sqrt{(t+\gamma)(\lambda-t)}}{\sqrt{(t+\beta)(1-t^2)(\mu-t)}} dt + 0$$

For integration with values of $t \leq 0$, replace t with -T; then the above equation

$$-i(s + d_2) = M \int_{-m}^{1} \frac{(m+T)\sqrt{(\gamma-T)(\lambda+T)}}{\sqrt{(\beta-T)(1-T^2)(\mu+T)}} dT + 0$$

or $i(s + d_2) = MI_4 = \frac{is}{I_1} I_4$ where $I_4 = \int_{-m}^{1} f_4(t)dt$

or $\frac{d_2+s}{s} = \frac{I_4}{I_1}$

or $\boldsymbol{F_3} = \frac{\boldsymbol{d_2+s}}{\boldsymbol{s}} - \frac{\boldsymbol{I_4}}{\boldsymbol{I_1}}$ ---- (3.5)

v. Integration between the vertices C and B ($-\gamma \le t \le -1$)

applying the conditions:

at point C: t = -1, for which z = 0 and

at point B: t = - γ, for which z = $-b_1$

we obtain:

$$0 = M\int_{-\gamma}^{-1}\frac{(m - t)\sqrt{(t+\gamma)(\lambda - t)}}{\sqrt{(t+\beta)(1- t^2)(\mu - t)}}dt - b_1$$

For integration with values of $t \le 0$, replacing with t with -T; then the above equation comes to:

$$0 = M\int_{1}^{\gamma}\frac{(m+T)\sqrt{(\gamma - T)(\lambda + T)}}{\sqrt{(\beta - T)(T^2 - 1)(\mu + T)}}\frac{dT}{i} - b_1$$

or $\quad 0 = MI_5 - b_1 = \frac{is}{I_1}.\frac{\boldsymbol{I_5}}{\boldsymbol{I_1}} - b_1 \qquad$ where $I_5 = \int_1^{\gamma} f_5(t)dt$

or $\quad \frac{b_1}{s} = \frac{I_5}{I_1}$

or $$\boldsymbol{F_4} = \frac{\boldsymbol{b_1}}{\boldsymbol{s}} - \frac{\boldsymbol{I_5}}{\boldsymbol{I_1}} \qquad \text{---- (3.6)}$$

vi. Integration between the vertices B and A ($-\beta \le t \le -\gamma$)

applying the conditions:

at point B: t = - γ, for which z = $-b_1$ and

at point A: t = - β, for which z = $-b_1 + id_1$

we obtain:

$$-b_1 = M\int_{-\beta}^{-\gamma}\frac{(m - t)\sqrt{(t+\gamma)(\lambda - t)}}{\sqrt{(t+\beta)(1- t^2)(\mu - t)}}dt - b_1 + id_1$$

For integration with values of $t \le 0$, replacing with t with -T; then the above equation comes to:

$$0 = M\int_{\gamma}^{\beta}\frac{(m+T)\sqrt{(\gamma - T)(\lambda + T)}}{\sqrt{(\beta - T)(1 - T^2)(\mu + T)}}dT + id_1$$

$$id_1 = M\int_{\beta}^{\gamma} \frac{(m+T)\sqrt{(T-\gamma)(\lambda+T)}}{\sqrt{(\beta-T)(T^2-1)(\mu+T)}}dT$$

or $\quad id_1 = MI_6 = \frac{is}{I_1}.I_6 \qquad$ where $I_6 = \int_{\beta}^{\gamma} f_6(t)dt$

or $\quad \frac{d_1}{s} = \frac{I_6}{I_1}$

or $\quad \boldsymbol{F_5} = \frac{\boldsymbol{d_1}}{\boldsymbol{s}} - \frac{\boldsymbol{I_6}}{\boldsymbol{I_1}} \qquad$ - - - - (3.7)

The parameters β, γ, m, λ and μ are to be found for known values of $\frac{d_1}{s}, \frac{d_3}{s}, \frac{d_2+s}{s}, \frac{b_1}{s}$ and $\frac{b_2}{s}$ from Eqs. (3.3) to (3.7). The equations are non-linear. Newton-Raphson technique has been used to find the solution. As the integrals are improper, the singularities have been removed by using the Gaussian-Quadrature method of substitution. The substitutions are carried out using the Jacobian Matrix as under.

$$\begin{bmatrix} \frac{\partial F_1^*}{\partial\beta} & \frac{\partial F_1^*}{\partial\gamma} & \frac{\partial F_1^*}{\partial m} & \frac{\partial F_1^*}{\partial\lambda} & \frac{\partial F_1^*}{\partial\mu} \\ \frac{\partial F_2^*}{\partial\beta} & \frac{\partial F_2^*}{\partial\gamma} & \frac{\partial F_2^*}{\partial m} & \frac{\partial F_2^*}{\partial\lambda} & \frac{\partial F_2^*}{\partial\mu} \\ \frac{\partial F_3^*}{\partial\beta} & \frac{\partial F_3^*}{\partial\gamma} & \frac{\partial F_3^*}{\partial m} & \frac{\partial F_3^*}{\partial\lambda} & \frac{\partial F_3^*}{\partial\mu} \\ \frac{\partial F_4^*}{\partial\beta} & \frac{\partial F_4^*}{\partial\gamma} & \frac{\partial F_4^*}{\partial m} & \frac{\partial F_4^*}{\partial\lambda} & \frac{\partial F_4^*}{\partial\mu} \\ \frac{\partial F_5^*}{\partial\beta} & \frac{\partial F_5^*}{\partial\gamma} & \frac{\partial F_5^*}{\partial m} & \frac{\partial F_5^*}{\partial\lambda} & \frac{\partial F_5^*}{\partial\mu} \end{bmatrix} \begin{bmatrix} \Delta\beta \\ \Delta\gamma \\ \Delta m \\ \Delta\lambda \\ \Delta\mu \end{bmatrix} = \begin{bmatrix} F_1 & (\beta^*,\gamma^*,m^*,\lambda^*,\mu^*) \\ F_2 & (\beta^*,\gamma^*,m^*,\lambda^*,\mu^*) \\ F_3 & (\beta^*,\gamma^*,m^*,\lambda^*,\mu^*) \\ F_4 & (\beta^*,\gamma^*,m^*,\lambda^*,\mu^*) \\ F_5 & (\beta^*,\gamma^*,m^*,\lambda^*,\mu^*) \end{bmatrix}$$

3.2 Mapping of ω-plane on to t-plane

The complex potential ω is defined as ω = ϕ + iψ in which,

ϕ = velocity potential and ψ = stream function.

The velocity potential function ϕ is defined as

$\phi = -k\left(\frac{p}{\gamma_w} + y\right) + C \qquad$ Where C is a constant. $\qquad$ - - - - (3.8)

The ω-plane for the flow domain of Figure 3.1(a) is shown in Figure 3.1(c).

If $C = k(h_2 - d_2 + d_3)$, then the velocity potential at downstream bed $\phi_d = 0$, and the velocity potential at upstream bed $\phi_u = -kh$

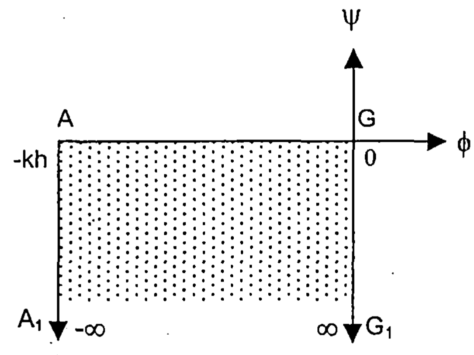


Figure .3.1(c) ω - plane

Mapping of the complex potential plane on to the t-plane is given by:

$$\omega = M_1 \int (t+\beta)^{-0.5} (\mu - t)^{-0.5} dt + N_1 \qquad \text{---- (3.9)}$$

$$or\ \omega = M_1 . Sin^{-1}\left(\frac{2t+\beta-\mu}{\beta+\mu}\right) + N_1 \qquad \text{---- (3.10)}$$

where M_1 and N_1 are complex constants.

Applying the conditions at the points A and G, in ω-plane and t-plane, the constants M_1 and N_1 are computed as under:

i. At point G ($\omega = 0$, $t = \mu$)
Substituting ω and t in the equation (3.10), we have:

$$0 = M_1 Sin^{-1}\left(\frac{2\mu+\beta-\mu}{\beta+\mu}\right) - N_1$$

or $\quad 0 = M_1 Sin^{-1}(1) + N_1$

or $\quad 0 = M_1 \frac{\pi}{2} + N_1$

or $\quad N_1 = -M_1 \frac{\pi}{2}$

ii. At point A ($\omega = -kh$, $t = -\beta$)
Substituting ω and t in the equation (3.10), we have:

$$-kh = M_1 Sin^{-1}\left(\frac{-2\beta+\beta-\mu}{\beta+\mu}\right) - M_1 \frac{\pi}{2}$$

or $\quad -kh = -M_1 \frac{\pi}{2} - M_1 \frac{\pi}{2}$

or $M_1 = \frac{kh}{\pi}$ - - - - (3.11)

and $N_1 = -\frac{kh}{2}$ - - - - (3.12)

putting the value of M_1 and N_1 in the equation (3.10), the complex potential is computed as: $\omega = \frac{kh}{\pi} Sin^{-1}\left(\frac{2t+\beta-\mu}{\beta+\mu}\right) - \frac{kh}{2}$ - - - - (3.13)

the equation (3.13) is the general equation, which provides relation between ω-plane and t-plane. For the boundary BCDEF, ψ = 0 and hence; ω = ϕ, so the equation (3.13) becomes:

$$\phi = \frac{kh}{2}\left\{\frac{2}{\pi} Sin^{-1}\left(\frac{2t+\beta-\mu}{\beta+\mu}\right) - 1\right\} \quad \text{- - - - (3.14)}$$

now equating the value of ϕ from equation (3.8), we have;

$-k\left(\frac{p}{\gamma_w} + y\right) + k(h_2 - d_2 + d_3) = \frac{kh}{2}\left\{\frac{2}{\pi} Sin^{-1}\left(\frac{2t+\beta-\mu}{\beta+\mu}\right) - 1\right\}$ which yields,

$$\boldsymbol{p = \gamma_w\left[h_2 - d_2 + d_3 - y - \frac{h}{2}\left\{\frac{2}{\pi} Sin^{-1}\left(\frac{2t+\beta-\mu}{\beta+\mu}\right) - 1\right\}\right]} \quad \text{- - - - (3.15)}$$

3.3 The pressure Distribution

The equation (3.15) is the general equation for seepage pressure under the floor. To find the pressure at various points below the floor, the ordinate 'y' from z-plane and the corresponding 't' from t-plane has to be entered in the equation (3.15).

i. At point A (y = d_1, t = - β)

$$p_A = \gamma_w h_1 \quad \text{- - - - (3.16)}$$

ii. At point B (y = 0, t = - γ)

$$\boldsymbol{p_B} = \gamma_w\left[h_2 - d_2 + d_3 - \frac{h}{2}\left\{\frac{2}{\pi} Sin^{-1}\left(\frac{-2\gamma+\beta-\mu}{\beta+\mu}\right) - 1\right\}\right] \quad \text{- - - - (3.17)}$$

iii. At point C (y = 0, t = - 1)

$$\boldsymbol{p_C} = \gamma_w\left[h_2 - d_2 + d_3 - \frac{h}{2}\left\{\frac{2}{\pi} Sin^{-1}\left(\frac{\beta-\mu-2}{\beta+\mu}\right) - 1\right\}\right] \quad \text{- - - - (3.18)}$$

iv. At point D (y = - (d_2+s), t = m)

$$\boldsymbol{p_D} = \gamma_w\left[h_2 + d_3 + s - \frac{h}{2}\left\{\frac{2}{\pi} Sin^{-1}\left(\frac{2m+\beta-\mu}{\beta+\mu}\right) - 1\right\}\right] \quad \text{- - - - (3.19)}$$

v. At point E (y = - d_2, t = 1)

$$\boldsymbol{p_E} = \gamma_w \left[h_2 + d_3 - \frac{h}{2}\left\{\frac{2}{\pi} Sin^{-1}\left(\frac{2+\beta-\mu}{\beta+\mu}\right) - 1\right\}\right] \qquad \text{- - - - (3.20)}$$

vi. At point F (y = - d_2, t = λ)

$$\boldsymbol{p_F} = \gamma_w \left[h_2 + d_3 - \frac{h}{2}\left\{\frac{2}{\pi} Sin^{-1}\left(\frac{2\lambda+\beta-\mu}{\beta+\mu}\right) - 1\right\}\right] \qquad \text{- - - - (3.21)}$$

vii. At point G (y = - (d_2 - d_3), t = μ)

$$\boldsymbol{p_G} = \gamma_w h_2 \qquad \text{- - - - (3.22)}$$

3.4 The Exit Gradient

The exit gradient (Harr 1962) is expressed as:

$$I_E = \frac{i}{k}\left(\frac{\partial w}{\partial t} \cdot \frac{\partial t}{\partial z}\right)$$

Using equations (3.1), (3.2) and (3.13) in the above we have;

$$I_E = \frac{hI_1}{\pi s}\left\{\frac{\sqrt{(t^2-1)}}{(t-m)\sqrt{(t+\gamma)(t\ -\ \lambda)}}\right\}$$

Maximum exit gradient occurs at ‘G’, where t = μ;

$$I_{Emax} = \frac{hI_1}{\pi s}\left\{\frac{\sqrt{(\mu^2-1)}}{(\mu-m)\sqrt{(\mu+\gamma)(\mu-\lambda)}}\right\} \qquad \text{- - - - (3.23)}$$

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4. CONFORMAL MAPPING TECHNIQUE

4.1 General

Most of the analytical method for solution of two-dimensional groundwater problems is concerned with the determination of a function, which will transform a problem from a geometrical domain within which a solution is sought for into the one within which the solution is known. The following narrations deals with the study of elementary functions and the manner in which these functions transform geometric figures from one complex plane to another.

Conformal Mapping Technique is a tool for solving two-dimensional Laplace equations. The method is used for solving the problems of flow under hydraulic structures.

4.2 Conformal Mapping Technique

It is generally known that for a weir with flat base and resting on the surface of ground, the stream lines or the lines of flow are confocal ellipses with their focci at 'O' as shown in the **Figure-A1**. The equation to these ellipses is given by: $\frac{x^2}{\left(\frac{b}{2}\text{Cosh u}\right)^2} + \frac{y^2}{\left(\frac{b}{2}\text{Sinh u}\right)^2} = 1$, where **u** is the streamline function.

Consider the physical domain in the Z-plane (**Figure-A2**). When a vertical obstruction like a sheet pile or the stepped depression is introduced, the configurations of the streamlines or the flow lines are distorted.

By applying the Scwartz-Christoffel transformation technique, the distortion can be brought back to normal configuration. The streamlines that will be formed after the transformation are smooth ellipses with confocal points.

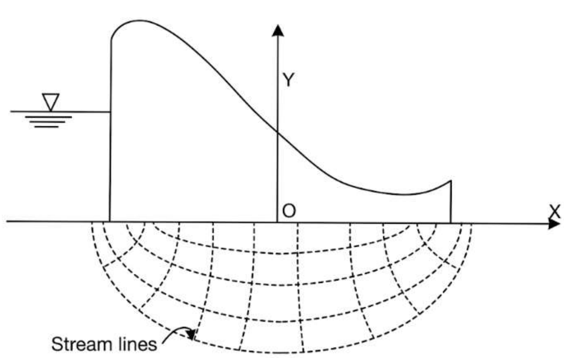

Figure-A1. Streamlines for flat base weirs on surface

Assuming the physical domain to be on the z-plane where any point on this is given by the relation Z = x + iy, the transformed plane is known as t-plane where any point on this is described by $\zeta = \xi + i\eta$.

In this process, the physical flow domains in z-plane as well as the complex potential domain ω are transformed on to a common platform known as the auxiliary **t-plane** from which a direct relation between the **z-plane** and **ω-plane** are obtained. As such, the flow region in the z-plane is first mapped on to the lower half of an auxiliary t-plane. Then the complex potential plane is also mapped on to the t-plane (**Figure-A3)**. From these two conformal mappings, the relationship between z and ω is obtained. The transformation is given by the relation:

$$Z = M\int \frac{dt}{(t-\alpha_1)^{\lambda_1}(t-\alpha_2)^{\lambda_2}(t-\alpha_3)^{\lambda_3}(t-\alpha_4)^{\lambda_4}(t-\alpha_5)^{\lambda_5}(t-\alpha_6)^{\lambda_6}(t-\alpha_7)^{\lambda_7}} \quad \ldots\ldots.(1)$$

Where $\lambda_1\pi$, $\lambda_2\pi$, $\lambda_3\pi$, $\lambda_4\pi$, $\lambda_5\pi$, $\lambda_6\pi$, $\lambda_7\pi$ are the changes in the angles at vertices at A, B, C, D, E, F, G in the positive sense and $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7$ are the ordinates at the points A, B, C, D, E, F, G in the t-plane on which the points A, B, C, D, E, F, G of the z-plane are mapped.

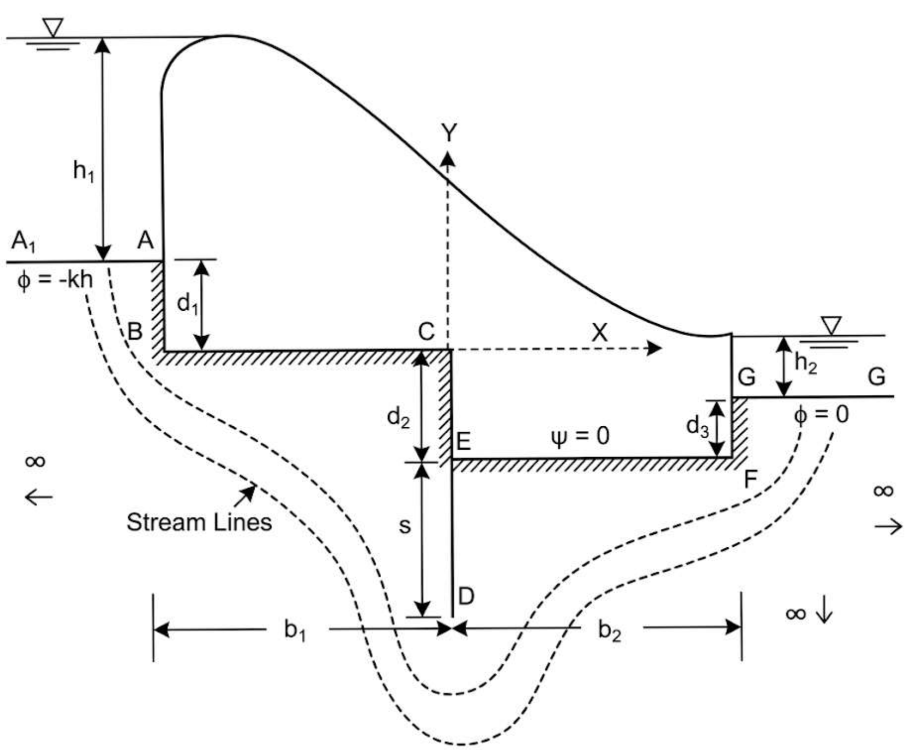

Figure-A2 Physical Domain in Z-plane

As seen (**Figure A2**) on z-plane, the angles of turning at A, B, C, D, E, F, G are $\frac{\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, -\pi, \frac{\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}$ respectively so that;

$\lambda_1\pi = \frac{\pi}{2}$ or $\lambda_1 = \frac{1}{2}$, $\lambda_2\pi = -\frac{\pi}{2}$ or $\lambda_2 = -\frac{1}{2}$, $\lambda_3\pi = \frac{\pi}{2}$ or $\lambda_3 = \frac{1}{2}$ and so on.

The origin in the figure in z-plane is at C, which has been chosen at a point midway between CE in t-plane.

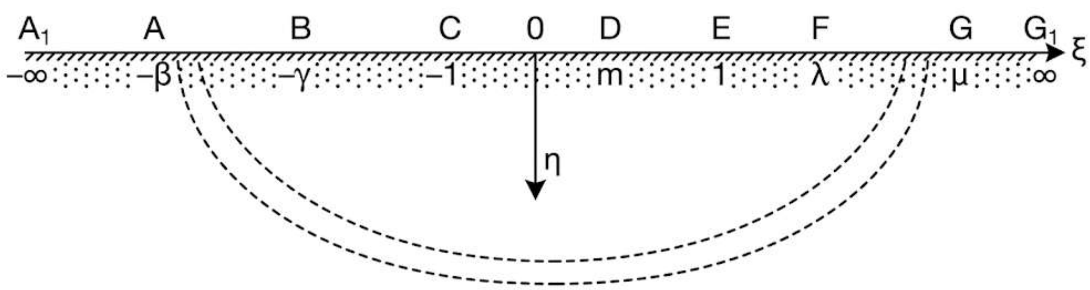

Figure-A3 Physical Domain Mapped on t-plane

Now assuming: $\alpha_1 = -\beta$ $\alpha_2 = -\gamma$ $\alpha_3 = -1$ $\alpha_4 = m$

$\alpha_5 = +1$ $\alpha_6 = \lambda$ $\alpha_7 = \mu$

the equation of transformation reduces to:

$$Z = M\int \frac{dt}{(t+\beta)^{½}(t+\gamma)^{-½}(t+1)^{½}(m\text{-}t)^{-1}(1\text{-}t)^{½}(\lambda\text{-}t)^{-½}(\mu\text{-}t)^{½}} + N \ldots\ldots. (1)$$

Or

$$Z = M\int \frac{(m\text{-}t)\sqrt{(t+\gamma)(\lambda\text{-}t)}}{\sqrt{(t+\beta)(1-t^2)(\mu - t)}}\,dt + N \ldots\ldots\ldots\ldots (2)$$

The above is the general equation for the relation between z-plane and t-plane obtained by Scwartz-Christoffel transformation technique for the physical domain shown in in **Figure A2**.

Similarly, by applying the same transformation technique; the relation between ω-plane and t-plane is obtained and explained in section 3.2 above.

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